**Symmetries and conservation laws**

The least action formulation is a convenient framework on which to examine the symmetries of the system. Symmetries imply the presence of conserved quantities as we’ll see, and the identification of symmetries in a Lagrangian to identify such conserved quantities is a technique prevalently used in quantum mechanics, and quantum field theory. First let’s state Emily Noether’s theorem.



**Temporal independence and conservation of energy**

Consider a Lagrangian, L(*q*i,i) independent of any *explicit* functional dependence on time, t [note that L will in general depend and change with time through the coordinates’ dependence on time, though]. Then ∂L/∂t = 0 [though dL/dt ≠ 0]. What is the conserved quantity associated with this? Let’s construct dL/dt to find out. So we have, using the chain rule:



and so we find,



This quantity we call the Hamiltonian, and so we see that if there is no explicit t dependence in L, H will be a conserved quantity.



What is H, physically? Suppose we have a Lagrangian like:



Then H is:



In other words, the conserved quantity is simply the energy, E.



I’ve been calling this conserved quantity H because it is known by another name – the Hamiltonian. Dynamics can be re-expressed in terms of this entity, instead of the Lagrangian and this affords a nice conceptual segue into quantum mechanics. We’ll take a look at the Hamiltonian later.

**Coordinate Symmetries**

Now let’s look at the conservation laws implied by some continuous coordinate symmetries, namely a process whereby making the change:



doesn’t change the Lagrangian. First, we’ll take advantage of the specification that it’s a continuous change by expanding f in a Taylor series about λ = 0. Then we’ll have:



where δqi is going to be, presumably, some function of the coordinates qj. Now let’s consider the change in the Lagrangian that results from this coordinate change. So we’ll have:



This new Lagrangian will involve new combinations of coordinates, and could in general result in new equations of motion for the individual qi’s. Expanding in a Taylor series:



and so we see that if this coordinate change doesn’t result in a Lagrangian change, i.e. δL = 0, then we have a conserved quantity, whose time derivative is zero too:



**Displacement Symmetry and Conservation of Momentum**

Consider a two particle Lagrangian with a two particle interaction:



We know that energy will conserved from our discussion above, because there is no explicit t-dependence. What about other symmetries? Well, if we move both particles by the same amount δ**r** = λ**a**, then the Lagrangian will be unchanged, since we’ll have:



So what conserved quantity is associated with this translational invariance? Well it’s



Now **a** is arbitrary, and so we see that it is the total momentum of the system that is conserved.



**Rotation symmetry and conservation of angular momentum**

What if our system is symmetric w/r to rotations? Consider again our two particle interaction L,



If we rotate the system by a constant amount δ**φ**, then still the Lagrangian should be preserved because we haven’t changed the velocities and we haven’t changed the potential energy. Let’s work out exactly how the coordinates change when we do this rotation. So to first order, we’d have (can justify this because we know that rotational velocity is **ω**×**r**, and so infinitesimal displacement would be dt(**ω**×**r**) = d**φ**×**r)**:



so then



So what is the conserved quantity associated with this?



Now δφ is arbitrary, so we see that the conserved quantity is just thte total angular momentum of the system,

